Pergamon

J. Appl. Maths Mechs, Vol. 60, No. 2, pp. 323–327, 1996

0021–8928/96 $24.00+0.00

S0021–8928(96)00041-X

NON-LINEAR STABILITY AND BIFURCATION OF THE FORK TYPE IN DYNAMICAL SYSTEMS WITH THE SIMPLEST SYMMETRY†

L. G. LOBAS

Kiev

(*Received* 14 *February* 1995)

Sufficiently smooth n-dimensional dynamical systems with odd right-hand sides are considered in which the simplest form of symmetry is obtained. An explicit expression for the first non-zero Lyapunov coefficient is obtained and invariant Poincaré manifolds are constructed for the case when the linearization matrix has a single non-zero eigenvalue and the real parts of the remaining eigenvalues are negative. A geometric illustration of loss of stability associated with creation bifurcation or bifurcation in the merge of steady states at a critical value of an essential parameter is presented for two-dimensional systems.

Explicit expressions for the first Lyapunov quantity in the case of two- and four-dimensional dynamical systems with quadratic non-linearities were obtained for the first time in [1]. Here, the question of stability in the critical case was associated with the nature of the safety of the boundary of the stability domain in the space of the parameters. The paper by Fishman [2] is also concerned with the latter problem. The local qualitative picture of the behaviour of dynamical systems close to the critical values of the parameters is known [4–11] in principle not only in classical cases [3] of a single non-zero root or a pair of pure imaginary roots of the characteristic equation but also in more complex cases. Results of a constructive character are also of significance from an applied point of view for calculating the governing quantities and their popularity is proportional to the simplicity of the mathematical apparatus required.

The systems

$$x = f(x,\upsilon), \quad x, f \in R^n, \quad \upsilon \in R_+^1 \tag{1}$$

are considered below which are invariant under the substitution $(t, x) \to (t, -x)$, that is, systems with an odd right-hand side with respect to the state variable x: $f(-x, \upsilon) = -f(x, \upsilon)$. In such systems the point $x = 0$ is necessarily the equilibrium state. Physically, this means that the system is indifferent to the direction of a displacement from the state $x = 0$. Duffing and Van der Pol oscillators as well as many means of transportation (cars, aircraft and ships) may serve as examples where the symmetry of the differential equations of the course motion means that left and right turns are equally possible. The principal parts of the non-linearities of system (1) in the neighbourhood of the state $x - 0$ are of the third order

$$\dot{x}_i = \sum_{j=1}^{n} a_{ij}x_j + P_i(x_1,\dots,x_n) \tag{2}$$

$$P_i = \sum_{k=1}^{n}\sum_{l=1}^{n}\sum_{m=1}^{n} a_{klm}^{(i)} x_k x_l x_m + o(|x|^3)$$

$$a_{ij} = \text{const}, \quad a_{klm}^{(i)} = a_{kml}^{(i)} = a_{mkl}^{(i)} = \text{const} \quad (i = 1,\dots,n)$$

We shall denote the eigenvalues of the matrix $A = \| a_{ij} \|_1^n$ by λ_1 and by A_{ij} the cofactor of the element a_{ij}. Let the point $O(0, \dots, 0) \in R^n$ be the precise state of equilibrium for which $\lambda_1 = 0$, Re $\lambda p < 0$ $(p = 2, \dots, n)$. We know that the linear part of system (2) admits of a linear integral with constant coefficients and we shall take it as one of the unknown functions $x = a_1x_1 + \dots + a_nx_n$. If the variables x_i are numbered in such a way that $A_{nn} \neq 0$, then $a_n \neq 0$ and x can be adopted instead of x_n.

Let us put $a_i = A_{in}$. Equations (2) become

$$x^{\cdot} = X(x, x_1, \dots, x_{n-1})$$

†*Prikl. Mat. Mekh.* Vol. 60, No. 2, pp. 327–332, 1996.

$$x_s^{\cdot} = \sum_{r=1}^{n-1} b_{sr}x_r + b_s x + P_s(x_1,\dots,x_{n-1},x_n(x,x_1,\dots,x_{n-1})) \tag{3}$$

$$X(x,x_1,\dots,x_{n-1}) = \sum_{i=1}^{n} a_i P_i(x_1,\dots,x_{n-1},x_n(x,x_1,\dots,x_{n-1}))$$

$$x_n(x,x_1,\dots,x_{n-1}) = \left(x - \sum_{s=1}^{n-1} a_s x_s \right) a_n^{-1}$$

$$b_{sr} = a_{sr} - b_r b_s, \quad b_s = a_{sn}a_n^{-1} \quad (s = 1,\dots,n-1)$$

We equate the right-hand sides of the non-critical equations to zero and seek the solution of the corresponding system of finite equations in the form $x_r = u_r(x)$, where $u_r(0) = 0$. The functions $u_r(x)$ are expanded in power series which converge for sufficiently small x. Such functions exist [3, 12]. Confining ourselves to quantities of the first order, we find

$$u_r = \sigma_r x + \dots, \quad \sigma_r = A_{nr}\left(\sum_{i=1}^{n} a_i A_{ni} \right)^{-1} \quad (r = 1,\dots,n-1)$$

In order to judge the stability of the zeroth solution of system (2), it is necessary to construct the expression

$$X(x,u_1(x),\dots,u_{n-1}(x)) = gx^3 + o(x^3)$$

From (3), we have

$$x_n(x,u_1(x),\dots,u_{n-1}(x)) = \sigma_n x + \dots$$

$$\sigma_n = \left(1 - \sum_{r=1}^{n-1} a_r \sigma_r \right) a_n^{-1}$$

Hence

$$g = \sum_{i=1}^{n} a_i \sum_{k=1}^{n} \sum_{l=1}^{n} \sum_{m=1}^{n} a_{klm}^{(i)} \sigma_k \sigma_l \sigma_m$$

If $g < 0$, the unperturbed motion is asymptotically stable [3] and the hypersurface $A_n = 0$ is a safe [1] boundary of the domain of stability in the space of the parameters. When $g > 0$, the unperturbed motion is unstable and the hypersurface $A_n = 0$ is a dangerous boundary in the direction $dA_n/d\upsilon < 0$ and the separation of a representative point takes place in it. Here, $A_n = |A|$ is the free term of the characteristic equation and υ is a characteristic parameter of the system such that $\upsilon \lesseqgtr \upsilon_+ \Rightarrow A_n \gtreqless 0$

We will now consider *the case of systems, the linear part of which is reduced to a basis of eigenvectors*. Let the spectrum of the matrix A be known and let the matrix A change in the following way when the parameter υ is changed: if $\upsilon < \upsilon_*$ then $\lambda_1 < 0, \dots, \lambda_s < 0$, $\lambda_{s+1} = \kappa_1 + i\omega_1$, $\lambda_{s+2} = \kappa_1 - i\omega_1, \dots, \lambda_{n-1} = \kappa_q + i\omega_q$, $\lambda_n = \kappa_q - i\omega_q$, $q = 1/2(n - s)$, $\kappa_k < 0$; if, however, $\upsilon > \upsilon_+$ then $\lambda_1 > 0$, the form of the eigenvalues $\lambda_2, \dots, \lambda_n$ is as before. Next, let $\upsilon = \upsilon_+$, that is, $\lambda_1 = 0$. We reduce system (2) to the form (summation over k, l and m)

$$\dot{\xi}_1 = \sum A_{klm}^{(1)} \xi_k \xi_l \xi_m + \dots \tag{4}$$

$$\dot{\xi}_2 = \lambda_2 \xi_2 + \sum A_{klm}^{(2)} \xi_k \xi_l \xi_m + \dots$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\dot{\xi}_s = \lambda_s \xi_s + \sum A_{klm}^{(s)} \xi_k \xi_l \xi_m + \dots$$

$$\dot{\xi}_{s+1} = \kappa_1 \xi_{s+1} - \omega_1 \xi_{s+2} + \sum A_{klm}^{(s+1)} \xi_k \xi_l \xi_m + \dots$$

$$\dot{\xi}_{s+2} = \omega_1 \xi_{s+1} + \kappa_1 \xi_{s+2} + \sum A_{klm}^{(s+2)} \xi_k \xi_l \xi_m + \dots$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\dot{\xi}_{n-1} = \kappa_q \xi_{n-1} - \omega_q \xi_n + \sum A_{klm}^{(n-1)} \xi_k \xi_l \xi_m + \dots$$

$$\dot{\xi}_n = \omega_q \xi_{n-1} + \kappa_q \xi_n + \sum A_{klm}^{(n)} \xi_k \xi_l \xi_m + \dots$$

where $A^{(i)}_{klm} = A^{(i)}_{kml} = A^{(i)}_{mkl}$. Since, the critical variable ξ_1 does not occur linearly in the equations for the non-critical variables $\xi_2, \ldots, \xi_n$, system (4) satisfies a well-known theorem [12, pp. 93, 94]. The first non-zero Lyapunov coefficient is therefore $g = A^{(1)}_{111}$. By finding the invariant manifold of system (4) in the form $\xi_j = \xi_j(\xi_1)$, $\xi_j(\xi_1) \equiv A_j\xi_1 + B_j\xi_1^3 + \ldots$ it can be shown that, in order to find it, it is necessary to equate the right-hand sides of the non-critical equations to zero. We obtain

$$\xi_\nu = -\frac{A^{(\nu)}_{111}}{\lambda_\nu}\xi_1^3 + \ldots \quad (\nu = 2, \ldots, s)$$

$$\xi_{s+1} = -\frac{A^{(s+1)}_{111}\varkappa_1 + A^{(s+2)}_{111}\omega_1}{\varkappa_1^2 + \omega_1^2}\xi_1^3 + \ldots$$

$$\xi_{s+2} = \frac{A^{(s+1)}_{111}\omega_1 - A^{(s+2)}_{111}\varkappa_1}{\varkappa_1^2 + \omega_1^2}\xi_1^3 + \ldots$$

....

$$\xi_{n-1} = -\frac{A^{(n-1)}_{111}\varkappa_q + A^{(n)}_{111}\omega_q}{\varkappa_q^2 + \omega_q^2}\xi_1^3 + \ldots$$

$$\xi_n = \frac{A^{(n-1)}_{111}\omega_q - A^{(n)}_{111}\varkappa_q}{\varkappa_q^2 + \omega_q^2}\xi_1^3 + \ldots$$

The behaviour of the one-dimensional system to which the problem of the stability of the zeroth solution of the n-dimensional system (4) is reduced is described by the equation

$$\dot{\xi}_1 = A^{(1)}_{111}\xi_1^3 + \ldots \tag{5}$$

In the case of two-dimensional systems it is possible to write out all the intermediate transformations in explicit form and to give a clear geometric illustration of the behaviour of the system. The invariant manifold is

$$\xi_2 = -A^{(2)}_{111}\lambda_2^{-1}\xi_1^3 + \ldots \tag{6}$$

Substituting Eq. (6) into the first equation of the system

$$\dot{\xi}_1 = \sum_{k=1}^{2}\sum_{l=1}^{2}\sum_{m=1}^{2} A^{(1)}_{klm}\xi_k\xi_l\xi_m + o(|\xi|^3) \tag{7}$$

$$\dot{\xi}_2 = \lambda_2\xi_2 + \sum_{k=1}^{2}\sum_{l=1}^{2}\sum_{m=1}^{2} A^{(2)}_{klm}\xi_k\xi_l\xi_m + o(|\xi|^3)$$

we obtain Eq. (5). In a certain finite neighbourhood of the origin of coordinates for the phase velocity vector of system (7) the approximation $V \approx \{A^{(1)}_{111}\xi^3{}_1, \lambda_2, \xi_2\}$ is permissible. Starting from this, it is possible to construct the phase fluxes and the cubic parabola (6) attracting these fluxes. The number of possible situations is equal to four and they correspond to the situations $A^{(1)}_{111} \gtrless 0, A^{(2)}_{111} \gtrless 0$.

In Fig. 1, the invariant manifold (6) is represented by the dot–dash curve and $A^{(2)}_{111} > 0$ in Fig. 1(a) and (b), $A^{(2)}_{111} < 0$ in Fig. 1(c) and (d), $A^{(1)}_{111} < 0$ in Fig. 1(a) and (c) and $A^{(1)}_{111} > 0$ in Fig. 1(b) and (d).

The phase fluxes are attracted by the manifold (6) and are then directed along it to the origin of coordinates or even depart from the latter. In the cases of Fig. 1(a) and (c), the origin of coordinates is a stable node of the non-linear system (7) while, in Fig. 1(b) and (c), it is a saddle, and curve (6) is a separatrix consisting of whiskers which emerge from the origin of coordinates.

Since $\lambda_1 < 0$, $\lambda_2 < 0$ when $\upsilon < \upsilon_+$, the origin of coordinates of the phase plane x_1x_2 is a stable node and its Poincaré index $j = 1$. When $\upsilon > \upsilon_+$, we have $\lambda_1 > 0$, $\lambda_2 < 0$, $j = -1$, that is, the point (0, 0) is a saddle.

A change in the sign of the Poincaré index of the origin of coordinates as a singular point of Eqs (2) when $n = 2$ or the equations

$$\dot{\xi}_i = \lambda_i\xi_i + \sum_{k=1}^{2}\sum_{l=1}^{2}\sum_{m=1}^{2} A^{(i)}_{klm}\xi_k\xi_l\xi_m + o(|\xi|^3) \quad (i = 1, 2)$$

can be explained either by a creation bifurcation or a bifurcation of the junction of singular points at the origin

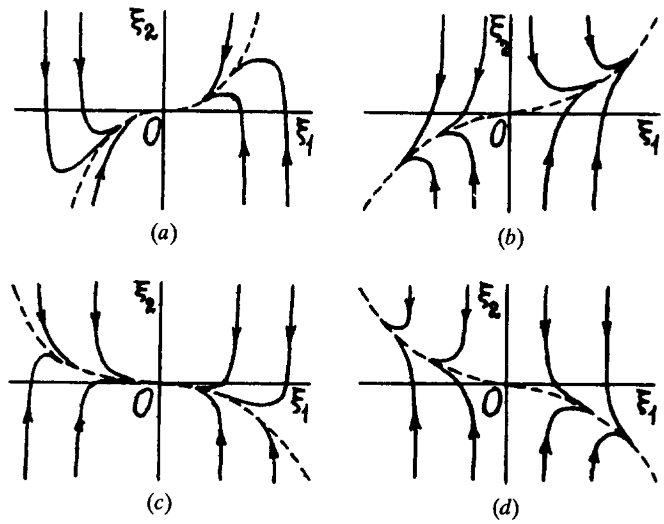

Fig. 1.

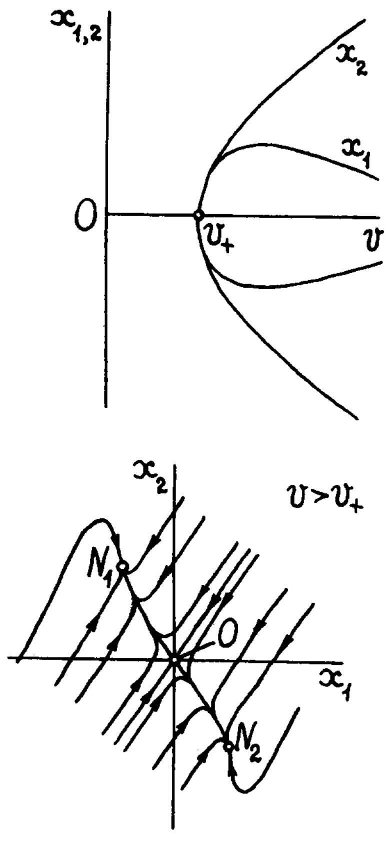

Fig. 2.

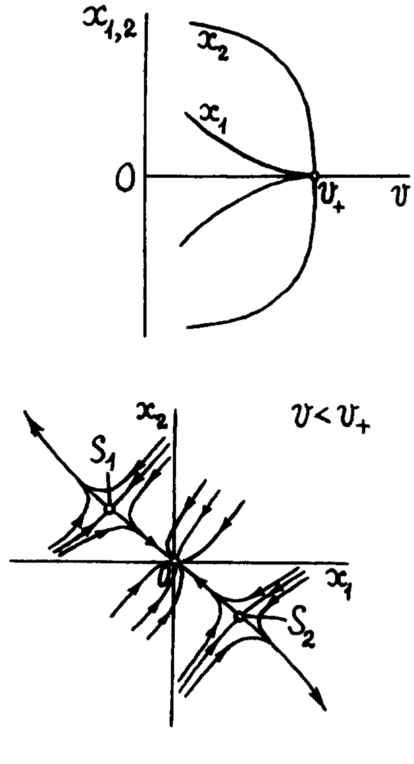

Fig. 3.

of coordinates when the parameter υ is changed. The first case is illustrated in Fig. 2 and the second case in Fig. 3.

In the case of creation bifurcation (generation) when $\upsilon < \upsilon_+$, there is only one singular point, the point (0, 0) in a finite neighbourhood of the origin of coordinates of the phase plane. When $\upsilon = \upsilon_+$ two new singular points are created at the point O such that, when $\upsilon > \upsilon_+$, apart from the singular point O, there are singular points N_1 and N_2 which are stable nodes. The subsequent perturbations for $\upsilon > \upsilon_+$ are bounded and the boundary $A_2 = 0$ of the stability domain in the space of the parameters of system (2) is safe.

In the case of merged bifurcation (annihilation) when $\upsilon < \upsilon_+$, there is a pair of saddle points S_1 and S_2 (Fig. 3), arranged symmetrically about the point O, which are stable nodes (in particular, a property of dynamical systems with symmetry is manifested in this). When the parameter υ increases, the points S_1 and S_2 approach the origin of coordinates, merging with it when $\upsilon = \upsilon_+$. In this case, the boundary $A_2 = 0$ is unsafe and separation of the representative point occurs in it.

The configuration of the curves in Figs 2 and 3 corresponds to a problem on the plane-parallel motion of an automobile within the framework of axiomatics [13] where υ is the longitudinal velocity of the centre of mass, $x_1 = \omega$ is the angular velocity of yaw and $x_2 = u$ is the lateral velocity of the centre of mass [14]. These curves synthesize the mechanism of the loss of stability of rectilinear motion (to which the point (0, 0) of the x_1x_2 plane corresponds) of an automobile with an excessive ability to tilt, which has been described in [15].

I wish to thank the referee for discussing the subject and content of this paper.

REFERENCES

1. BAUTIN N. N., *Behaviour of Dynamical Systems Close to Boundaries of the Stability Domain*. Nauka, Moscow, 1984.
2. FISHMAN L. Z., Criteria for safe and unsafe boundaries of the stability domain of systems with hysteresis in the case of a zero root. *Differents. Urav.* **26**, 10, 1830–1832, 1990.
3. LYAPUNOV A. M., *Collected Papers in Three Volumes*, Vol. 2. Izd. Akad. Nauk SSSR, Moscow, 1956.
4. ARNOL'D V. I., *Supplementary Chapters in the Theory of Ordinary Differential Equations*. Nauka, Moscow, 1978.
5. ARNOL'D V. L., AFRAIMOVICH V. S., IL'YASHENKO Yu. S. and SHIL'NIKOV Yu. S., The theory of bifurcations. In *Itogi Nauki i Tekhniki, Sovr. Probl. Mat. Fundam. Napravleniya*. VINITI, Moscow, 5, 5–218, 1982.
6. ARNOL'D V. I., *Catastrophe Theory*. Nauka, Moscow, 1990.
7. NEIMARK Yu. I., *Dynamical Systems and Control Processes*. Nauka, Moscow, 1978.
8. HOLODNIOK M., KLICVA A., KUBICVEK M. and MAREK M., *Methods of Analysing Non-linear Dynamical Models*. Mir, Moscow, 1991.
9. TROGER H. and STEINDL A., *Nonlinear Stability and Bifurcation Theory*. Springer, Vienna, 1991.
10. BAZYKIN A. D., KUZNETSOV Yu. A. and KHIBNIK A. I., *Bifurcation Diagrams of Dynamical Systems in a Plane*. Nauchn. Tsentr Biol. Issledovanii, Akad. Nauk SSSR, Pushchino, 1985.
11. BAZYKIN A. D., KUZNETSOV Yu. A. and KHIBNIK A. I., *Bifurcation Patterns*. Znaniye, Moscow, 1989.
12. MALKIN I. G., *Theory of Stability of Motion*. Nauka, Moscow, 1966.
13. ROCARD Y., *L'instabilité en Mécanique. Automobiles. Avions. Point Suspendus*. Masson, Paris, 1954.
14. LOBAS L. G. and VERBITSKII V. G., *Qualitative and Analytic Methods in the Dynamics of Wheeled Machines*. Naukova Dumka, Kiev, 1990.
15. VERBITSKII V. G. and LOBAS L. G., A method of determining singular points and their character in the problem of the plane motion of a wheeled carriage. *Prikl. Mat. Mekh.* **45**, 5, 944–948, 1981.

Translated by E.L.S.